JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 1, January–February 1994

Engineering Notes

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Some Conservation Laws for Orbits Involving Variable Mass and Linear Damping

L. Cveticanin*
Faculty of Technical Science,
Novi Sad 21000, Yugoslavia

Introduction

IN this Note the extension in dynamic analysis of the systems with variable mass under the influence of the central force and damping is considered. The task is to find some of the conservation laws. Noether's theorem[1] defines the condition for existence of a conservation law of the dynamic system. The system considered in this Note is a Lagrangian nonconservative dynamic system. The conservation law exists for every infinitesimal transformation of the generalized coordinates and time that leaves Hamilton's action integral absolute or gauge-invariant. Let us assume the space and time generators F^i and f, respectively, which are supposed to be functions of the generalized coordinates x^i, generalized velocities $\dot{x}^i$, and time t. The conservation law

$$\left(\frac{\partial L}{\partial \dot{x}^i}\right)F^i + \left[L - \left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i\right]f - P(t, x, \dot{x}) = C = \text{const} \tag{1}$$

exists, when Noether's identity

$$\left(\frac{\partial L}{\partial x^i}\right)F^i + \left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{F}^i + \left[L - \left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i\right]\dot{f} + \left(\frac{\partial L}{\partial t}\right)f - \dot{P}(t, x, \dot{x}) = 0 \tag{2}$$

is satisfied.

The complexity of the problem is connected with the fact that Noether's theorem does not offer any suggestions for obtaining the conservation laws. The solution of this problem leads to a system of first-order partial differential equations that we call generalized Killing's equations.

Extended Kepler Problem

The differential equation of motion of the body with variable mass under influence of a central force and damping is

$$m(t)\left(\frac{d^2\boldsymbol{r}}{dt^2}\right) = m(t)k\boldsymbol{r}r^{n-1} + \boldsymbol{\phi} - \delta\left(\frac{d\boldsymbol{r}}{dt}\right) \tag{3}$$

where $m(t)$ is the variable mass, $\boldsymbol{r}$ the position vector to the unmovable point, r the magnitude of the position vector, k the coefficient of the central force, $n = 0, \pm 1, \pm 2, \ldots$, δ a damping coefficient and $\boldsymbol{\phi}$ the reactive force. In the general case the relative velocity of the particle dm is described as a fraction of the velocity $(d\boldsymbol{r}/dt)$. Then the reactive force is

$$\boldsymbol{\phi} = \left(\frac{dm(t)}{dt}\right)\left(\frac{p}{q}\right)\left(\frac{d\boldsymbol{r}}{dt}\right) \tag{4}$$

where p/q is a positive or negative quantity. For the polar coordinates the differential equations of motion are

$$m(t)(\ddot{r} - r\dot{\theta}^2) = m(t)kr^n + \left(\frac{dm(t)}{dt}\right)\left(\frac{p}{q}\right)\dot{r} - \delta\dot{r} \tag{5a}$$

$$m(t)(2\dot{r}\dot{\theta} + r\ddot{\theta}) = \left(\frac{dm(t)}{dt}\right)\left(\frac{p}{q}\right)r\dot{\theta} - \delta r\dot{\theta} \tag{5b}$$

where r and θ are polar coordinates. The second equation gives the first integral of motion

$$r^2\dot{\theta}m(t)^{-p/q}\exp\left(\delta\int\frac{dt}{m(t)}\right) = K = \text{const} \tag{6}$$

Introducing $\dot{\theta}$ from Eq. (6) into Eq. (5a) it is

$$\ddot{r} - K^2m^{2(p/q)}r^{-3}\exp\left(-2\delta\int\frac{dt}{m}\right) = kr^n + \left(\frac{dm(t)}{dt}\right)\left(\frac{p}{q}\right)\frac{\dot{r}}{m} - \frac{\delta\dot{r}}{m} \tag{7}$$

[For simplicity, $m(t) \equiv m$].

Despite the fact that the system is nonconservative, a Lagrangian function exists and is of the form

$$L = \dot{r}^2m^{-p/q}\exp\left(\delta\int\frac{dt}{m}\right)\Big/2 - K^2m^{p/q}r^{-2}\exp\left(-\delta\int\frac{dt}{m}\right)\Big/2 + kr^{n+1}m^{-p/q}\exp\left(\delta\int\frac{dt}{m}\right)\Big/(n+1) \tag{8}$$

Let us suppose the space and time generators and the gauge function have the forms

$$F = F(t, r), \qquad f = f(t, r), \qquad P = P(t, r) \tag{9}$$

Substituting Eq. (8) and Eq. (9) into Eq. (2) and equating to zero terms of the various powers of $\dot{r}$ the following system of Killing equations results:

$$\dot{r}^0\colon \left[K^2m^{p/q}r^{-3}\exp\left(-\delta\int\frac{dt}{m}\right) + kr^nm^{-p/q}\exp\left(\delta\int\frac{dt}{m}\right)\right]F - \left(\frac{\partial f}{\partial t}\right)\left[K^2m^{p/q}r^{-2}\exp\left(-\delta\int\frac{dt}{m}\right)\Big/2 + kr^{n+1}m^{-p/q}\exp\left(\delta\int\frac{dt}{m}\right)\Big/(n+1)\right]$$
$$+ f\left[-K^2m^{(p/q)-1}r^{-2}\left(\frac{p}{q}\right)\left(\frac{dm}{dt}\right)\exp\left(-\delta\int\frac{dt}{m}\right)\Big/2\right.$$

Received May 11, 1992; revision received Nov. 30, 1992; accepted for publication Dec. 22, 1992.

*Professor, Faculty of Technical Sciences, V. Vlahovica 3.

$$+K^2 m^{(p/q)-1} r^{-2} \delta \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2\Bigg] + kr^{n+1} m^{-(p/q)-1}$$

$$\times \left(\frac{-p}{q}\right)\left(\frac{dm}{dt}\right) \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1) - \frac{\partial P}{\partial t} = 0 \tag{10a}$$

$$\dot{r}: \left(\frac{\partial F}{\partial t}\right) m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right) - \left(\frac{\partial f}{\partial r}\right)\Bigg[K^2 m^{p/q} r^{-2}$$

$$\times \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2 + kr^{n+1} m^{-(p/q)} \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1)\Bigg]$$

$$-\left(\frac{\partial P}{\partial r}\right) = 0 \tag{10b}$$

$$\dot{r}^2: \Bigg[m^{-(p/q)-1}\left(\frac{-p}{q}\right)\left(\frac{dm}{dt}\right) \exp\left(\delta \int \frac{dt}{m}\right)\Big/2$$

$$+ m^{-(p/q)-1} \delta \exp\left(\delta \int \frac{dt}{m}\right)\Big/2\Bigg] f$$

$$-\left(\frac{\partial f}{\partial t}\right) m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right)\Big/2$$

$$+\left(\frac{\partial F}{\partial r}\right) m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right) = 0 \tag{10c}$$

$$\dot{r}^3: \left(\frac{\partial f}{\partial r}\right) m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right) = 0 \tag{10d}$$

The last equation implies that $f = f(t)$. Supposing the absolute invariance of the action integral, that is, $P = 0$, Eq. (10b) indicates that $F = F(r)$ (since $\partial F/\partial t = 0$). The remaining Eqs. (10a) and (10c) are reduced to the following system

$$\left[K^2 m^{p/q} r^{-3} \exp\left(-\delta \int \frac{dt}{m}\right) + kr^n m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right)\right] F$$

$$-\left(\frac{df}{dt}\right)\Bigg[K^2 m^{p/q} r^{-2} \exp\left(-\delta \int \frac{dt}{m}\Big/2 + kr^{n+1} m^{-p/q}\right.$$

$$\left.\times \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1)\right)\Bigg] + f\Bigg[-K^2 m^{(p/q)-1} r^{-2}\left(\frac{p}{q}\right)$$

$$\times \left(\frac{dm}{dt}\right) \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2\Bigg] + K^2 m^{(p/q)-1} r^{-2}$$

$$\times \delta \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2 + kr^{n+1} m^{-(p/q)-1}\left(\frac{-p}{q}\right)$$

$$\times \left(\frac{dm}{dt}\right) \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1) = 0 \tag{11a}$$

$$fm^{-1}\left[\left(\frac{-p}{q}\right)\left(\frac{dm}{dt}\right) + \delta\right]\Big/2 - \left(\frac{df}{dt}\right)\Big/2 + \left(\frac{dF}{dr}\right) = 0 \tag{11b}$$

The space generator is assumed to have the form

$$F = \beta r \tag{12}$$

where β is an arbitrary constant different from zero.

Substituting Eq. (12) into Eq. (11b) we obtain the time generator

$$f = 2\beta m^{-(p/q)} \exp\left(\delta \int \frac{dt}{m}\right) \int \left[m^{p/q} \exp\left(-\delta \int \frac{dt}{m}\right)\right] dt \tag{13}$$

Let us introduce both generators into Eq. (11a). The connection between the damping coefficient δ and degree of the central force n as a function of mass variation law m is obtained

$$(n+3)/4 = \left[\left(\frac{p}{q}\right)\left(\frac{dm}{dt}\right) - \delta\right] m^{-(p/q)-1}$$

$$\times \exp\left(\delta \int \frac{dt}{m}\right) \int \left[m^{p/q} \exp\left(-\delta \int \frac{dt}{m}\right)\right] dt \tag{14}$$

According to Eq. (1) the corresponding conservation law is

$$m^{-(p/q)} \exp\left(\delta \int \frac{dt}{m}\right)\Bigg\{r\dot{r} + 2\Bigg[-\dot{r}^2 m^{-p/q} \exp\left(\delta \int \frac{dt}{m}\right)\Big/2$$

$$-K^2 m^{p/q} r^{-2} \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2 + kr^{n+1} m^{-p/q}$$

$$\times \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1)\Bigg] \int \left[m^{p/q} \exp\left(-\delta \int \frac{dt}{m}\right)\right] dt\Bigg\}$$

$$= C = \text{const} \tag{15}$$

We must emphasize that the conservation law is valid for $m(t)$ satisfying the restriction (14).

A special case of motion of the body with variable mass is of interest: the absolute velocity of the particle dm is zero (Levi-Civita case), considered in the following section.

Expanded Levi-Civita Equation

Levi and Civita assumed that the absolute velocity of the mass particle dm is zero.[2] For such an assumption the differential equation of motion of the body with variable mass in the central force field is simplified as

$$\frac{d}{dt}\left(m(t)\frac{dr}{dt}\right) = m(t)krr^{n-1} - \frac{\delta\, dr}{dt} \tag{16}$$

Using the previous consideration, two first integrals are obtained:

$$r^2\dot{\theta} m \exp\left(\delta \int \frac{dt}{m}\right) = K = \text{const} \tag{17}$$

and

$$m \exp\left(\delta \int \frac{dt}{m}\right)\Bigg\{r\dot{r} + 2\Bigg[-\dot{r}^2 m \exp\left(\delta \int \frac{dt}{m}\right)\Big/2$$

$$-K^2 m^{-1} r^{-2} \exp\left(-\delta \int \frac{dt}{m}\right)\Big/2 + kr^{n+1} m$$

$$\times \exp\left(\delta \int \frac{dt}{m}\right)\Big/(n+1)\Bigg] \int \left[m^{-1} \exp\left(-\delta \int \frac{dt}{m}\right)\right] dt\Bigg\}$$

$$= C = \text{const} \tag{18}$$

for

$$-(n+3)/4 = \left[\frac{dm}{dt} - \delta\right] \exp\left(\delta \int \frac{dt}{m}\right)$$

$$\times \int \left[m^{-1} \exp\left(-\delta \int \frac{dt}{m}\right)\right] dt \tag{19}$$

These two integrals are not enough to integrate the equation of motion but can help us to comment on the physical sense of the problem. Two examples follow.

Example 1. Let us assume the case when the mass variation is a linear function of time

$$m = m_0(1 - \alpha t) \tag{20}$$

where α is a constant value. According to Eq. (19) the damping coefficient must satisfy the relation

$$\delta = 4\alpha/(1+n) \tag{21}$$

When the central gravitation force is inversely proportional to the square of the distance $n = -2$

$$\delta/\alpha = 4/3 \tag{22}$$

and the conservation law is

$$(r\dot{r} + \dot{r}^2 m + 2kr^{-1}m)m^{-1/3} + 3K^2 m^{4/3} r^{-2}/(4\alpha) = C = \text{const} \tag{23}$$

Example 2. The damping coefficient satisfies the relation

$$\delta = -\left(\frac{\mathrm{d}m}{\mathrm{d}t}\right) \tag{24}$$

The trajectory is described in the form

$$\ddot{r} - K^2 r^{-3} = kr^n \tag{25}$$

The first integral is

$$\dot{r}^2 - 2kr^{n+1}/(n+1) + K^2 r^{-2} = \text{const} \tag{26}$$

It corresponds to the results in Refs. 3 and 4.

Conclusion

For the system with variable mass under the influence of a central nonlinear force and linear damping, conservation laws exist. The mass and the reactive force produced by mass variation are functions of time. The system presents a nonconservative system that has a Lagrangian. According to Noether's theorem such a system has conservation laws. In this Note the conservation laws are found. These first integrals have a very important role in considering the dynamic behavior and detecting chaotic motion of the system. The conservation laws presented in this Note are valuable only for strictly defined mass variation. For some simplifications in the system the standard Kepler problem is obtained and the conservation law reduces to the well-known form.

Acknowledgments

The investigations were supported by the government of Serbia, and VANU. It is my great pleasure to express at this point my gratitude to Dj. S. Djukic, who suggested this problem to me and provided me with very helpful advice and guidance.

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Evaluation of Missile Seeker Dwell Time for Three-Dimensional Aerial Engagements

C. B. Asthana* and Prahlada†

Research Centre Imarat, Hyderabad 500269, India

Nomenclature

A = circle obtained in plane 1 at the start of detection
a = position of target at the start of detection measured along the radius of A (i.e., along r)
R = range of target from missile at the start of detection (measured along line of sight)
r = radius of A
V = magnitude of $\bar{V}$ (i.e., $|\bar{V}|$)
$\bar{V}$ = relative velocity vector of target with reference to missile
$\bar{V}_n$ = $V \cos\gamma$
$\bar{V}_r$ = $V \sin\gamma$
β = $\tan^{-1}[r/\sqrt{(R^2 - a^2)}]$ (i.e., ½ beamwidth)
γ = smaller angle between $\bar{V}$ and plane 1 measured in plane 2
ϕ = angle between the outward radial vector of A containing the target and $\bar{V}_n$, measured in plane 1 (+ve anticlockwise as seen from the missile)
ψ = ½ the angle subtended at the antenna center in plane 2 by the chord of A along $\bar{V}_n$ (i.e., $\angle NOQ$, Fig. 2)

Introduction

SURFACE-to-surface missiles or air-to-air missiles that use mid-course guidance followed by a terminal homing phase need to acquire the target at the start of the homing guidance phase. For acquisition, the onboard radar system would require the target to be in the beam for a finite time. This finite dwell time on target is necessary to process the received signal for detection, confirmation, and subsequent tracking of the target. Hence the problem of determining a circular area perpendicular to the beam axis that accounts for a finite dwell time is important in the context of target acquisition without search in a three-dimensional aerial engagement. Kuno et al.[1] have considered the interaction of the radar beam with a plane that contains the target velocity vector. The intersection results in an ellipse, but for analytical convenience they have approximated it to a circle and pointed out that the error thus introduced would not affect the results for their application. Kouba and Bose[2] have evaluated the pointing angle error circular error probability (CEP) as a function of a unit pointing vector. The pointing angle error that results from the combined effect of all of the error sources is obtained as the root-sum-square of the individual error. Their analysis indicated the relation between the probability of target sighting by the missile and the error in the pointing vector. Kuno et al. and Kouba and Bose, however, did not address dwell time determination procedures. In this Note a three-dimensional engagement is considered and the evaluation of the minimum value of dwell time is represented in the form of a nomogram. Some of the variables are redefined more appropriately here, as compared with those given in Ref. 3. The variables that affect the estimate of the minimum dwell time have been grouped into an optimal set of independent parameters, namely V/R, a/r, γ, and β. The present study is not concerned with determining the seeker dwell time required for detection and confirmation of the tar-

Received June 29, 1992; revision received Jan. 10, 1993; accepted for publication Jan. 18, 1993.

*Scientist, Control System Laboratory.

†Project Director.